1) Permutation operators (optional): The permutation operator $\hat{P}_A$ associated with the permutation

$$A \longleftrightarrow \begin{pmatrix} 1 & 2 & \cdots & N \\ a_1 & a_2 & \cdots & a_N \end{pmatrix}$$

acts in the following way: $\hat{P}_A |x_1, \ldots, x_N\rangle = |x_{\bar{a}_1}, \ldots, x_{\bar{a}_N}\rangle$, where

$$A^{-1} \longleftrightarrow \begin{pmatrix} 1 & 2 & \cdots & N \\ \bar{a}_1 & \bar{a}_2 & \cdots & \bar{a}_N \end{pmatrix}.$$

a) Prove that the permutation operator $\hat{P}$ is unitary.

b) Show that for a hamiltonian $\hat{H}$ of a system of identical particles, and a permutation operator $\hat{P}$, the following relation holds: $\hat{P}\hat{H}\hat{P}^{-1} = \hat{H}$.

c) Show that if $|\psi\rangle$ is a nondegenerate eigenket of a hamiltonian $H$ for a system of identical particles, then $|\psi\rangle$ is either symmetric or antisymmetric under all pairwise exchanges.

2) Noninteracting particles: A single-particle quantum mechanical system possesses a Hilbert space spanned by three orthonormal eigenkets. Three particles occupy these states. How many distinct physical states are there if the three particles are:

a) Three identical fermions?

b) Three identical bosons?

c) Two identical fermions and a boson?

d) Two identical bosons and a fermion?

e) Three distinguishable fermions?

f) Three distinguishable bosons?

3) Identical spin-3/2 fermions: Consider 3 identical spin-3/2 fermions, for which spin and orbital degrees of freedom are not coupled. How many independent energy eigenstates are there associated with an orbital wave function $\psi(r_1, r_2, r_3)$ that is totally symmetric under permutation of its arguments?
4) Identical particles: Determine the conditions under which the following hamiltonians describe identical particles:

a) Two spin-1/2 degrees of freedom with

\[ \hat{H} = \sum_{\mu,\nu=1}^{3} \hat{S}^{(1)}_\mu \Delta_{\mu\nu} \hat{S}^{(2)}_\nu, \]

where \( \Delta_{\mu\nu} \) are the arbitrary complex elements of a rank-2 tensor.

b) Three spin-1 bosons with

\[ \hat{H} = \sum_{i=1}^{3} \frac{|\hat{p}^{(i)}|^2}{2m^{(i)}} + \sum_{i=1}^{3} \gamma^{(i)} \hat{S}^{(i)} \cdot \mathbf{B}^{(i)}(\hat{r}^{(i)}) + \sum_{1 \leq i < j \leq 3} W^{(ij)}(\hat{r}^{(i)}, \hat{r}^{(j)}). \]

5) Exchange interaction: In this question we will examine the exchange interaction, introduced in class. Consider two electrons in an atom and neglect (i) spin-orbit coupling for each electron, and (ii) the electron-electron interaction. Suppose that the particles occupy the two orbitals \( \phi_1(\mathbf{r}) \) and \( \phi_2(\mathbf{r}) \). Electrons are spin-1/2 particles: to each one we associate a spin observable \( \hat{S} \).

a) By including both spin and spatial degrees of freedom, build a basis of the possible physical states.

b) The electron-electron interaction \( \hat{U} \) is spin-independent and translationally invariant. What does this tell us about its matrix elements?

c) By treating the electron-electron interaction to first order in perturbation theory, show that an effective hamiltonian for this set of states takes the form

\[ \hat{H} = A \hat{I} - \frac{J}{2} \left( \hat{I} + \frac{4}{\hbar^2} \hat{S}^{(1)} \cdot \hat{S}^{(2)} \right). \]

What are \( A \) and \( J \) in terms of the orbital functions and the electron-electron interaction?

d) Now consider the four spin functions alone, and the action on them of the operator

\[ \frac{1}{2} \left( 1 + \frac{4}{\hbar^2} \hat{S}^{(1)} \cdot \hat{S}^{(2)} \right). \]

Why is this operator called the exchange operator?
6) **Real valued vectors and tensors:** In this question we are going to consider the real-valued vectors and tensors with which you are familiar. However we shall use a notation which should help to illuminate the notation we have been using for Hilbert spaces in quantum mechanics.

Consider a \(d\)-dimensional linear vector space. Normally we would write an arbitrary vector as \(t\). It is a linear combination of unit vectors, 
\[
  t = \sum_{\mu=1}^{d} t_\mu e_\mu,
\]
where \(\{e_\mu\}\) is an orthonormal set of basis vectors and 
\[
  e_\mu \cdot e_\nu = \delta_{\mu\nu}.
\]
Let us call this linear vector space \(G_1\).

Simply change the notation:
\[
  t \rightarrow |t\rangle, \quad \text{an arbitrary vector;}
\]
\[
  e_\mu \rightarrow |\mu\rangle, \quad \text{a basis vector;}
\]
\[
  t \cdot s = \langle t|s\rangle, \quad \text{an inner product.}
\]

Now consider the tensor 
\[
  \sigma = \sum_{\mu\nu} \sigma_{\mu\nu} e_\mu \otimes e_\nu.
\]
The set of tensors \(G_2\) is spanned by the basis \(\{e_\mu \otimes e_\nu\}\). Instead, we shall write 
\[
  |\sigma\rangle = \sum_{\mu\nu} \sigma_{\mu\nu} |\mu\rangle \otimes |\nu\rangle = \sum_{\mu\nu} \sigma_{\mu\nu} |\mu, \nu\rangle.
\]
Scalar products are defined by 
\[
  (e_\mu \otimes e_\nu) \cdot (e_\rho \otimes e_\tau) = (\langle \mu| \otimes \langle \nu|) (|\rho\rangle \otimes |\tau\rangle) = \langle \mu|\rho\rangle \langle \nu|\tau\rangle = \delta_{\mu\rho} \delta_{\nu\tau}.
\]

a) Evaluate \(\langle \sigma|\omega\rangle\) in terms of the components \(\sigma_{\mu\nu}\) and \(\omega_{\mu\nu}\).
b) How many real numbers are required to parametrise elements of \(G_2\)?
c) Symmetric tensors are the elements of \(G_2\) for which \(\sigma_{\mu\nu} = \sigma_{\nu\mu}\). How many real numbers are required to parametrise an arbitrary symmetric tensor?
d) How many elements are there in a basis for the symmetric tensors?
e) Write down a basis for the symmetric tensors for the case \(d = 3\) in terms of the tensors \(|\mu, \nu\rangle\).
f) Show that an arbitrary element of \(G_2\) can be written as the sum of two pieces, one symmetric and one antisymmetric. Can this be done for an arbitrary element of \(G_3\), where \(G_3\) is defined as the obvious extension of \(G_1\) and \(G_2\)?