1) **Uncorrelated (or unentangled) states**: Two identical bosons or fermions are in the normalised state

$$|\Psi\rangle = A \sum_{ij} c_i d_j a_i^\dagger a_j^\dagger |0\rangle,$$

where $i$ labels an orthonormal set of single-particle states, $\{c_i\}$ and $\{d_i\}$ are complex constants parametrising $|\Psi\rangle$, and $\{a_i^\dagger\}$ and $\{a_i\}$ are creation and annihilation operators for the single-particle states. Such a state is said to be uncorrelated (or unentangled) except for the effect of statistics.

a) Suppose that the complex numbers $\{c_i\}$ and $\{d_i\}$ satisfy

$$\sum_i |c_i|^2 = \sum_i |d_i|^2 = 1.$$

Determine the normalisation constant $A$ in terms of the sum $S = \sum_i c_i^* d_i$.

b) Compute the expectation value, in the state $|\Psi\rangle$, of a one-particle operator $\sum_a f^{(a)}$ in terms of the constants $\{c_i\}$ and $\{d_i\}$, and the matrix elements $\langle i | f | j \rangle$.

c) Show that if $S = 0$ then this one-particle expectation value has the same result as if the particles were distinguishable and occupied two certain one-particle states (not necessarily basis states $|i\rangle$). Which are these certain states?

d) Now consider a two-particle operator $V$ that is diagonal in the $|i\rangle$-basis (which means that $(ij|V|kl) = V_{ij} \delta_{ik} \delta_{jl}$). Calculate the expectation value of this operator in the state $|\Psi\rangle$ in terms of $\{c_i\}$, $\{d_i\}$ and $V_{ij}$. Show that the result is the same as for distinguishable particles, provided that $c_i d_i = 0$ for all $i$. Think about what this means when $\{|i\rangle\}$ is the position basis $\{|r\rangle\}$. 

2) **Boson coherent states (optional):** This question introduces a family of states known as boson coherent states. Consider a system with one eigenstate $\phi(\xi)$. As there is only one state, there is only one creation operator $a^\dagger$ and one annihilation operator $a$. Define the so-called coherent state $|z\rangle$, in which $z$ is an arbitrary complex number, via

$$|z\rangle \equiv \exp(za^\dagger)|0\rangle.$$ 

a) Show that $|z\rangle$ is an eigenket of $a$ with eigenvalue $z$.

b) Show that $\langle z'|z\rangle = \exp((z')^*z)$.

A string of creation and annihilation operators is said to be *normal-ordered operator* if all the creation operators occur to the left of all the annihilation operators. Given an un–normal-ordered operator, we construct its normal-ordered counterpart by re-ordering the string of creation and annihilation operators so that the annihilation operators all lie to the right of the creation operators. For any operator $f(a^\dagger, a)$, the normal-ordered counterpart is denoted $:f(a^\dagger, a):$.

c) Write down :$a a^\dagger:$.. Show that for coherent states we have

$$\langle z'| :f(a^\dagger, a): |z\rangle = f((z')^*, z) \exp((z')^*z),$$

where $f(z', z)$ is any function that is analytic in both arguments.

d) Show that the identity operator may be written in the form

$$I = \frac{1}{\pi} \int d \text{Re}z \ d \text{Im}z \ e^{-z^*z} |z\rangle\langle z|.$$ 

e) Show that $\{|z\rangle\}$ is overcomplete.

Consider a single quantum-mechanical harmonic oscillator with coordinate $q$, momentum $p$, and hamiltonian

$$H = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2q^2 - \frac{1}{2}\hbar\omega.$$ 

f) At time $t = 0$ the oscillator is prepared in the coherent state $|z_i\rangle$. Determine the state $|\Psi(t)\rangle$ at the subsequent time $t$. Does a coherent state remain coherent?

g) Compute the expectation values of $q$ and $p$ in the state $|\Psi(t)\rangle$.

h) Compare the expectation values obtained in part (i) with those obtained in energy eigenstates of the oscillator. Compare the time-dependence of the coherent-state expectation values of $q$ and $p$ with those of the classical oscillator.

i) Compute the uncertainties in $q$ and $p$ at time $t$. Comment on their product.

j) Compare the coherent state with the oscillator ground state.
3) **Field operators for bosons:** This question introduces so-called field operators for bosons. Recall that to define creation and annihilation operators we need a complete solution to a one-body eigenproblem, i.e., a set of eigenfunctions $\psi_i(\xi)$, with eigenvalues $\epsilon_i$. Then we

i) build the Hilbert space $\mathcal{I}_N$;

ii) symmetrise to obtain the boson space $\mathcal{B}_N$;

iii) collect together the boson spaces $\mathcal{B}_N$ and the vacuum space $\mathcal{B}_0$ to obtain the boson Fock space.

Creation and annihilation operators act on the occupation number representation of this Fock space in the following way:

$$a_j^\dagger |\ldots , n_j, \ldots \rangle = \sqrt{n_j + 1} |\ldots , n_j + 1, \ldots \rangle,$$

$$a_j |\ldots , n_j, \ldots \rangle = \sqrt{n_j} |\ldots , n_j - 1, \ldots \rangle.$$ 

Introduce the following linear combination of creation and annihilation operators, the so-called field operators:

$$\hat{\psi}^\dagger (\xi) = \sum_j \psi_j^*(\xi) a_j^\dagger,$$

$$\hat{\psi}(\xi) = \sum_j \psi_j(\xi) a_j.$$ 

Think of these simply as sets of operators parametrised by $\xi$.

a) By recalling that $\psi_i(\xi)$ form a complete orthonormal set of functions for $\mathcal{I}_1$, prove that

$$[\hat{\psi}(\xi), \hat{\psi}^\dagger (\xi')] = \delta (\xi - \xi').$$

b) Show that for spinless noninteracting bosons in a potential $U(r)$ the hamiltonian may be written in terms of field operators in the following way:

$$H = -\frac{\hbar^2}{2m} \int dr \, \hat{\psi}^\dagger (\mathbf{r}) \nabla^2 \hat{\psi}(\mathbf{r}) + \int dr \, \hat{\psi}^\dagger (\mathbf{r}) \hat{\psi}(\mathbf{r}) U(\mathbf{r}).$$

c) Describe in physical terms the effect of applying $\hat{\psi}^\dagger (\mathbf{r})$ to a state. Determine is the physical quantity to which the operator $\hat{\psi}^\dagger (\mathbf{r}) \hat{\psi}(\mathbf{r})$ corresponds. Determine the physical quantity to which the operator $\int d\mathbf{r} \, \hat{\psi}^\dagger (\mathbf{r}) \hat{\psi}(\mathbf{r})$ corresponds.

d) Find the Heisenberg equation of motion for the operator $\hat{\psi}(\mathbf{r})$ by evaluating the relevant commutators for the hamiltonian of part (b).

e) Give a physical interpretation to the amplitude

$$\langle G | \hat{\psi}(\mathbf{r}_2, t_2) \hat{\psi}^\dagger (\mathbf{r}_1, t_1) |G \rangle,$$

where the time-dependence indicates that the operators are in the Heisenberg representation, $|G\rangle$ is the ground state of the system, and you may assume that $t_2 > t_1$. 

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4) Nondegenerate perturbation theory: This question concerns first-order nondegenerate perturbation theory, couched in the language of the occupation number representation. Consider a system of identical particles (fermions or bosons) with hamiltonian

$$H = \sum_i \epsilon_i a_i^\dagger a_i + \frac{1}{2} \sum_{qrst} a_q^\dagger a_r^\dagger a_s a_t \langle qr|V|ts \rangle.$$ 

Show that the expectation value of $H$ in the occupation number state $|n_1, n_2, \ldots \rangle$ is given by

$$E_{n_1, n_2, \ldots} = \sum_i \epsilon_i n_i + \frac{1}{2} \sum_{q \neq r} n_q n_r \{\langle qr|V|qr \rangle \pm \langle qr|V|rq \rangle \} + \frac{1}{2} \sum_q n_q (n_q - 1) \langle qq|V|qq \rangle,$$

where the $+(-)$ sign holds for bosons (fermions). The two matrix elements in the second term on the right hand side of this equation are known as the direct and exchange terms, respectively.